Tensor Factorization via Matrix Factorization

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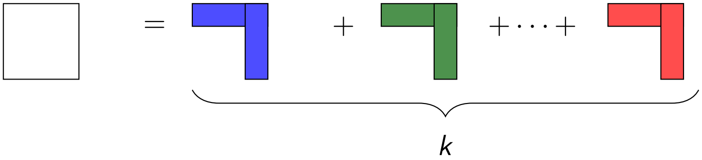
May 8, 2015

What is tensor (CP) factorization?

(Kolda and Bader 2009)

- Tensor analogue of matrix eigen-decomposition.

$$M = \sum_{i=1}^{k} \pi_i u_i \otimes u_i \qquad .$$



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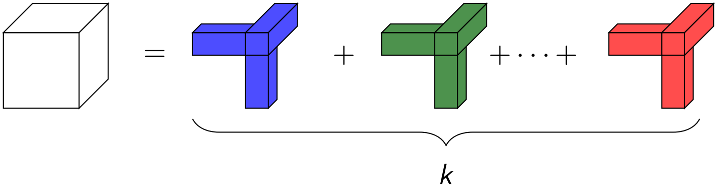

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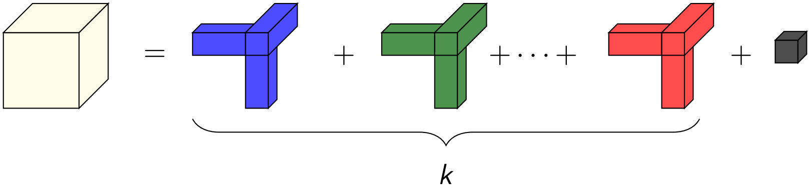
$$\widehat{T} = \sum_{i=1}^{k} \pi_i u_i \otimes u_i \otimes u_i + \epsilon R.$$

- **Goal:** Given T with noise, ϵR, **recover factors** u_i.



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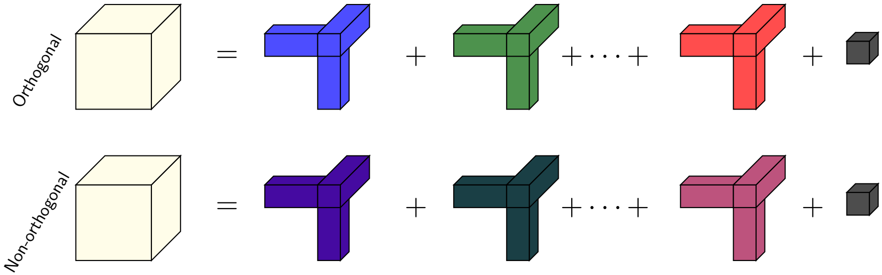

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- **Learning latent variable graphical models**
  - Anandkumar et al. 2013b
  - TODO: crowdsourcing
  - TODO: others

Existing tensor factorization algorithms

- **Tensor power method** (Anandkumar et al. 2013b)
  - Analog of matrix power method.
  - Sensitive to noise.
  - Restricted to orthogonal tensors.

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- **Both operate on the tensor directly.**

Our approach

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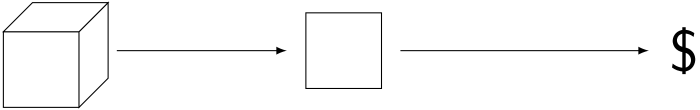
- **Approach:** use existing fast and robust **matrix** algorithms.



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- Simultaneous matrix factorizations
- Oracle projections
- Random projections

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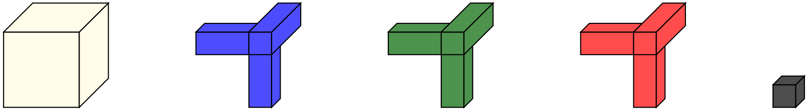
Tensor factorization via single matrix factorization



$$T = \pi_1 u_1^{\otimes 3} + \pi_2 u_2^{\otimes 3} + \pi_3 u_3^{\otimes 3} + \epsilon R$$

Tensor factorization via single matrix factorization

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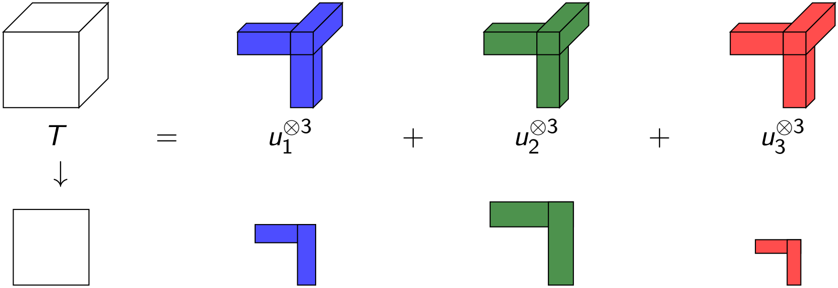
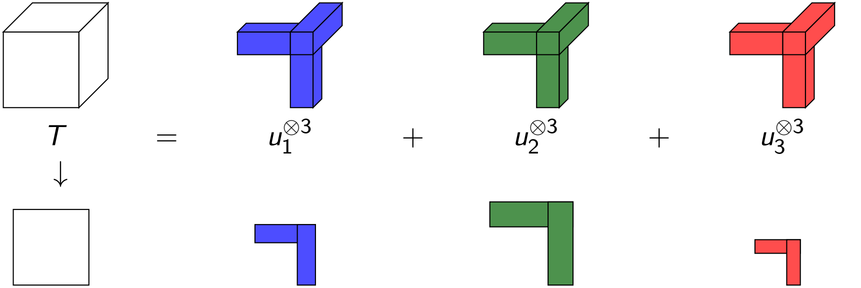


$$T = u_1^{\otimes 3} + u_2^{\otimes 3} + u_3^{\otimes 3}$$

$$\downarrow$$

$$T(I, I, w) = (w^\top u_1) u_1^{\otimes 2} + (w^\top u_2) u_2^{\otimes 2} + (w^\top u_3) u_3^{\otimes 3}$$

Tensor factorization via single matrix factorization



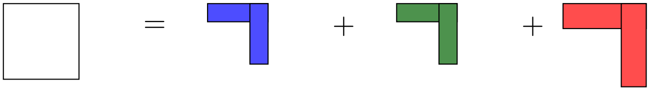
$$T = u_1^{\otimes 3} + u_2^{\otimes 3} + u_3^{\otimes 3}$$

$$\downarrow$$

$$T(I, I, w) = \underbrace{(w^\top u_1)}_{\lambda_1} u_1 u_1^\top + \underbrace{(w^\top u_2)}_{\lambda_2} u_2 u_2^\top + \underbrace{(w^\top u_3)}_{\lambda_3} u_3 u_3^\top$$

- **Proposal:** Eigen-decomposition on the projected matrix.

Sensitivity of single matrix projection



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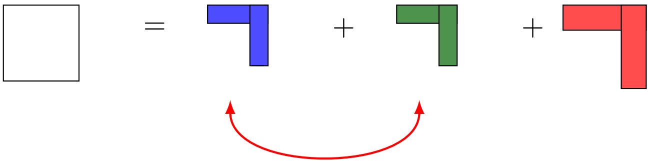

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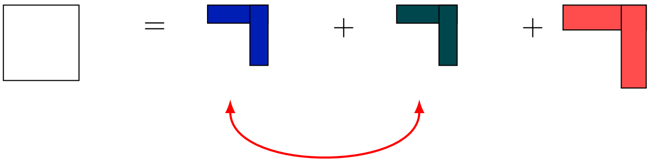

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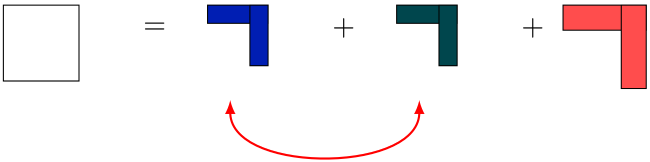
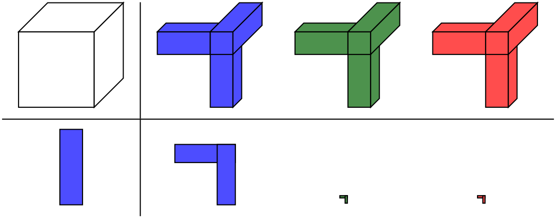


- If two eigenvalues are equal, corresponding eigenvectors are arbitrary.
- **Problem**: Eigendecomposition is very sensitive to the **eigengap**.

$$\text{error in factors} \propto \frac{1}{\min(\text{difference in eigenvalues})}.$$

Projections matter



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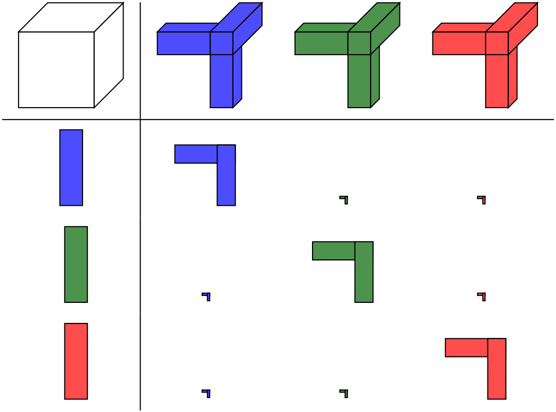

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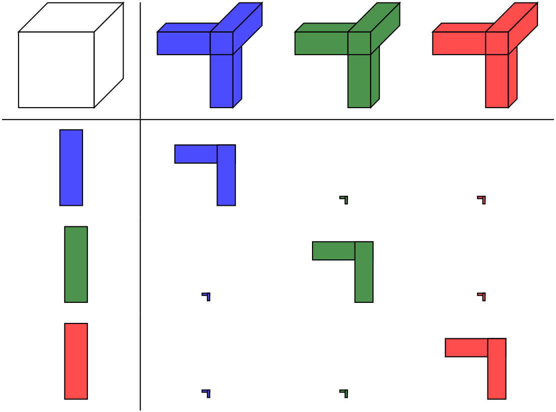

- **How can we leverage multiple projections?**

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Enter simultaneous diagonalization

$$\underbrace{T(I, I, w_1)}_{M_1} = \underbrace{(w_1^\top u_1)}_{\lambda_{11}} u_1 u_1^\top + \underbrace{(w_1^\top u_2)}_{\lambda_{21}} u_2 u_2^\top + \underbrace{(w_1^\top u_3)}_{\lambda_{31}} u_3 u_3^\top$$

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- **Projections share factors.**

Algorithm

- **Algorithm:** Simultaneously diagonalize projected matrices.

$$\widehat{U} = \arg\max_{\widehat{U}} \sum_{l=1}^{L} \text{off}(U^\top M_l U) \qquad \text{off}(A) = \sum_{i \neq j} A_{ij}^2.$$

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- Optimize using the Jacobi angles (Cardoso and Souloumiac 1996).
- Multiple projections proposed in Anandkumar, Hsu, and Kakade 2012, but didn't use simultaneous diagonalization.

Comparison with single matrix factorization

- Single matrix factorization depends on **minimum eigengap**.

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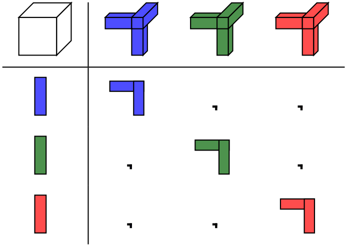
Pick k projections along the factors (u_i). Then,



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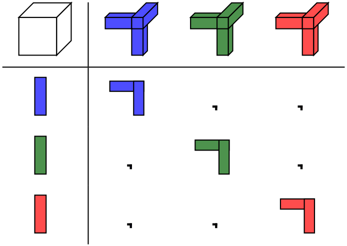

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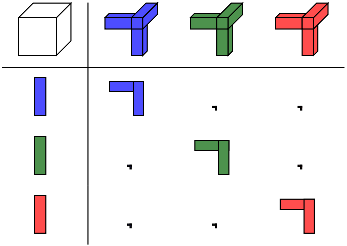

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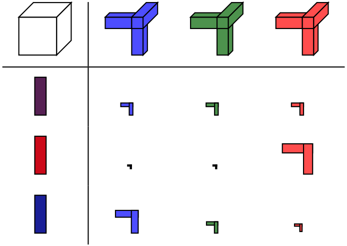

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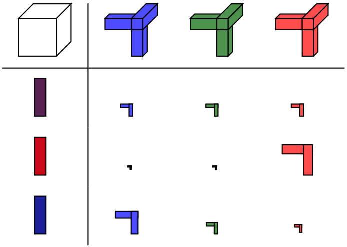

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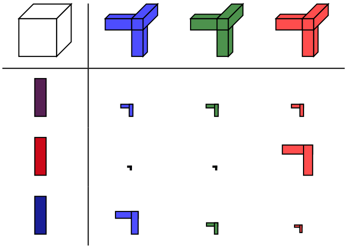

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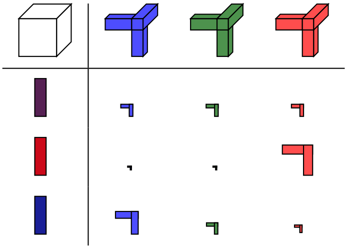

- As good as having oracle projections!

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  - Project tensor on to approximated factors.
  - Return factors $\tilde{u}_i$ from simultaneous diagonalization.

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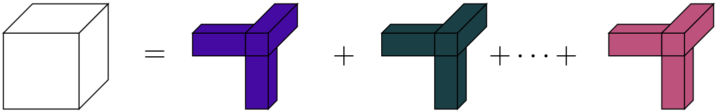
Naive approach: whitening non-orthogonal factors



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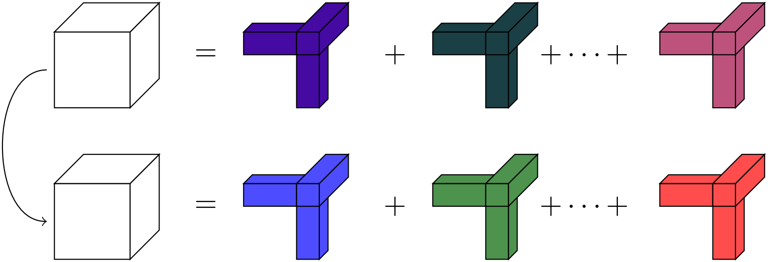

- Use a whitening transformation to orthogonalize tensor (Anandkumar et al. 2013b).

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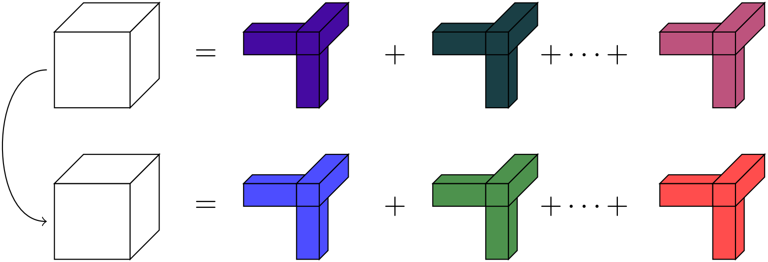

- Use a whitening transformation to orthogonalize tensor (Anandkumar et al. 2013b).
  - Is a major source of errors itself (Souloumiac 2009).

Non-orthogonal simultaneous diagonalization

$$\underbrace{T(I, I, w_1)}_{M_1} = \underbrace{(w_1^\top u_1)}_{\lambda_{11}} u_1 u_1^\top + \underbrace{(w_1^\top u_2)}_{\lambda_{21}} u_2 u_2^\top + \underbrace{(w_1^\top u_3)}_{\lambda_{31}} u_3 u_3^\top$$

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- **Note:** λ_{iI} are factor weights, not eigenvalues.

Non-orthogonal simultaneous diagonalization

- **Algorithm:** Simultaneously diagonalize projected matrices.

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- U are not constrained to be orthogonal.
- Optimize using the QR1JD algorithm (Souloumiac 2009).
  - Only guaranteed to have local convergence.

Results: Non-orthogonal simultaneous diagonalization

Theorem (Oracle projections)

Pick k projections along the factors (u_i). Then,

$$\textit{error in factors} \leq O\left(\|U^{-\top}\|_2^3 \frac{\sqrt{\pi_{\max}}}{\pi_{\min}^2} \right) \epsilon,$$

where $U = [u_1 | \cdots | u_k]$.

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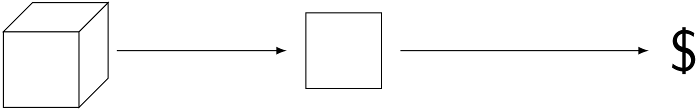

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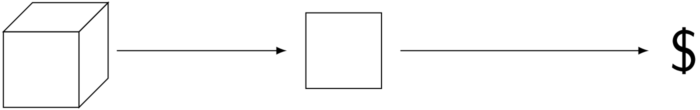

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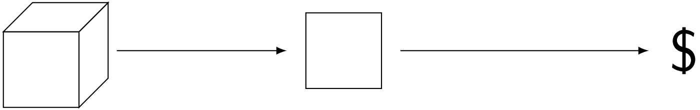

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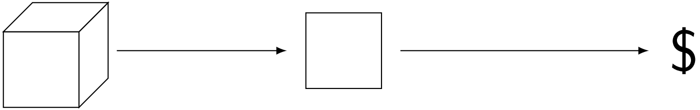
- **Robust** to noise with general support for non-orthogonal factors.
- **Competitive empirical** performance.

Conclusions



- Reduce tensor problems to matrix ones with $\tilde{O}(k)$ random projections.
- **Robust** to noise with general support for non-orthogonal factors.
- **Competitive empirical** performance.
- **Questions?**