

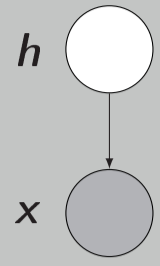


Spectral Experts for Estimating Mixtures of Linear Regressions

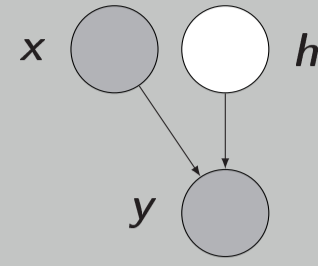
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Parameter Estimation in Latent Variable Models



Generative Models



Discriminative Models

- Latent variable models (LVMs) are hard to learn because latent variables introduce non-convexities in the log-likelihood function.
- In practice, local methods (EM, gradient descent, etc.) are employed, but these can stuck in local optima.
- Can we develop efficient consistent estimators for discriminative latent variable models?**
 - Why discriminative LVMs? Easy to add features, often more accurate.
 - The method of moments has been used for consistent parameter estimation in several generative LVMs, e.g. HMMs¹, LDA¹, and stochastic block models².
 - Can we extend these techniques to discriminative LVMs?
- Main result:** Consistent estimator for a simple discriminative model; the mixture of linear regressions.
 - Key Idea:** Expose tensor factorization structure using regression.
 - Theory:** We prove polynomial sample and computational complexity.

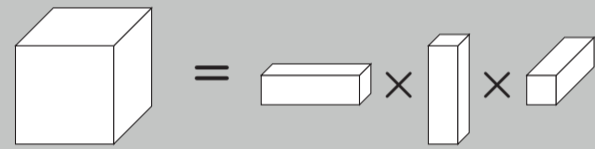
[1] Anandkumar, Hsu, Kakade, 2012; [2] Anandkumar, Ge, Hsu, Kakade, 2012

Aside: Tensor Operations

Tensor Product

$$x^{\otimes 3} = x \otimes x \otimes x$$

$$x_{ijk}^{\otimes 3} = x_i x_j x_k$$



Inner product

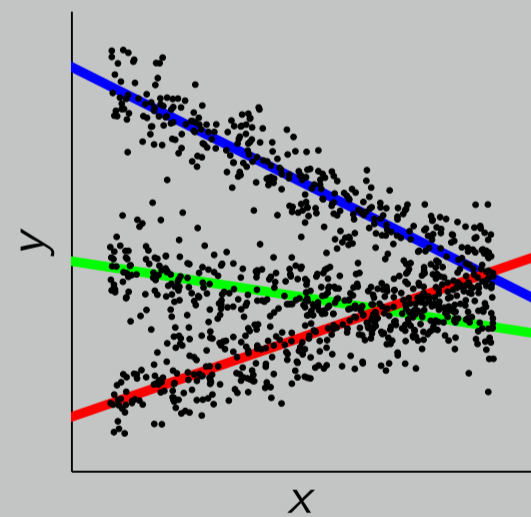
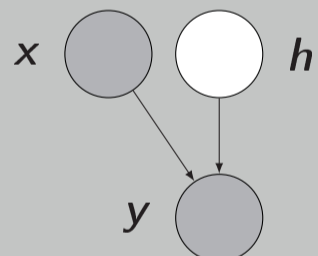
$$\langle A, B \rangle = \sum_{ijk} A_{ijk} B_{ijk}$$

$$= \langle \text{vec } A, \text{vec } B \rangle$$

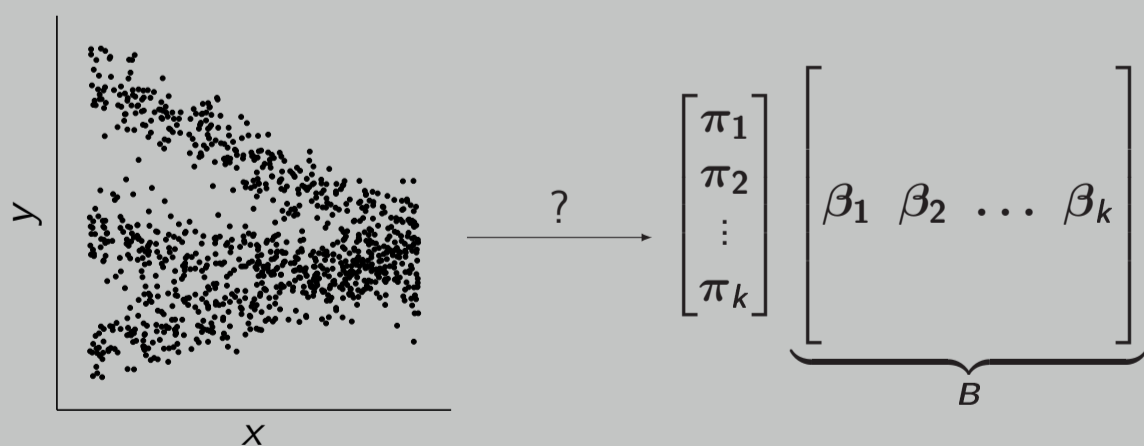
$$\langle \text{cube}, \text{cube} \rangle = 0.5$$

$$\langle \text{vector}, \text{vector} \rangle = 0.5$$

Mixture of Linear Regressions



- For a particular x , we draw y as follows,
 - $h \sim \text{Mult}([\pi_1, \pi_2, \dots, \pi_k])$.
 - $y = \beta_h^T x + \epsilon$.
- Given $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$, we want to recover the parameters π and B .



- Our approach uses low-rank regression to reduce the problem to tensor eigendecomposition.



Sample Complexity: $O(k \|x\|^{12} \|\beta\|^6 \mathbb{E}[\epsilon^2])$ $O\left(\frac{k \pi_{\max}^2}{\sigma_k(M_2)^5}\right)$

Step 1: Finding Tensor Structure via Regression

- Key Observation:** Regression on the powers of (y, x) gives us the expected powers of the regression coefficients β .

$$y = \langle \beta_h, x \rangle + \epsilon$$

$$= \underbrace{\langle \mathbb{E}[\beta_h], x \rangle}_{\text{linear measurement}} + \underbrace{(\beta_h - \mathbb{E}[\beta_h])^T x + \epsilon}_{\text{noise}}$$

$$y^2 = (\langle \beta_h, x \rangle + \epsilon)^2$$

$$= \underbrace{\langle \mathbb{E}[\beta_h^{\otimes 2}], x^{\otimes 2} \rangle}_{M_2} + \text{bias}_2 + \text{noise}_2$$

$$y^3 = \underbrace{\langle \mathbb{E}[\beta_h^{\otimes 3}], x^{\otimes 3} \rangle}_{M_3} + \text{bias}_3 + \text{noise}_3$$

- M_2 and M_3 are both of rank k , so we can use low rank regression^{3,4}!

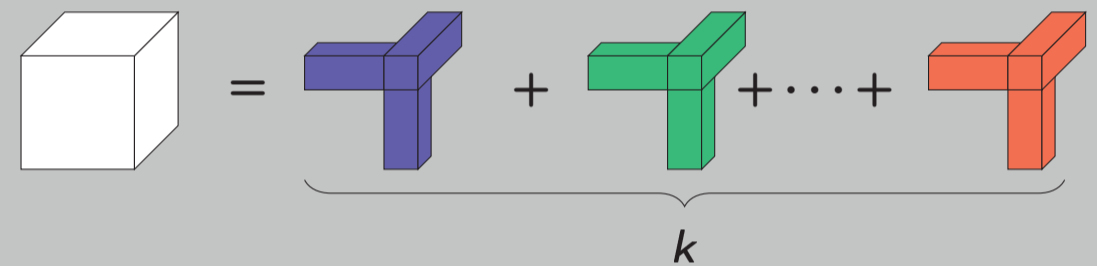
$$\hat{M}_2 = \arg \min_M \sum_{(x,y) \in \mathcal{D}} (y^2 - \langle M, x^{\otimes 2} \rangle - \text{bias}_2)^2 + \lambda_2 \|M\|_*$$

$$\hat{M}_3 = \arg \min_M \sum_{(x,y) \in \mathcal{D}} (y^3 - \langle M, x^{\otimes 3} \rangle - \text{bias}_3)^2 + \lambda_3 \|M\|_*$$

[3] Fazel, 2002; [4] Tomoika, Hayashi and Kashima, 2010

Step 2: Parameter Recovery via Tensor Factorization

- M_3 has a low-rank tensor decomposition: $M_3 = \sum_{h=1}^k \pi_h \beta_h^{\otimes 3}$

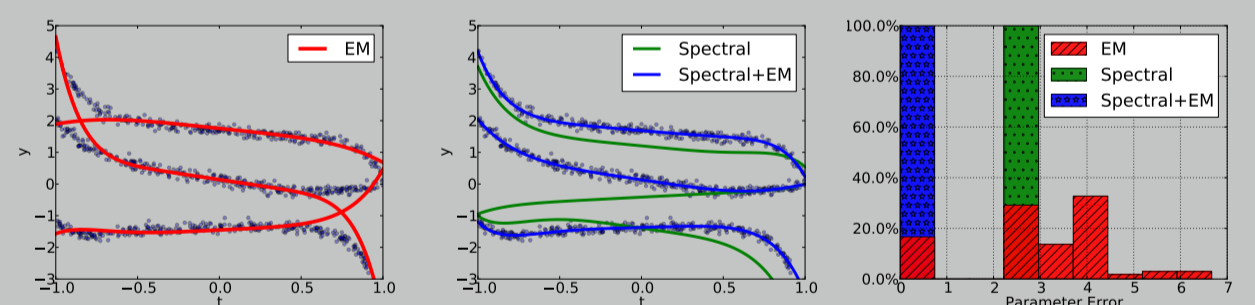


- Key Observation:** If β_h are orthogonal, they are eigenvectors⁵; $M_3(\beta_h, \beta_h) = \pi_h \beta_h$.
- In general, we can whiten M_3 first.

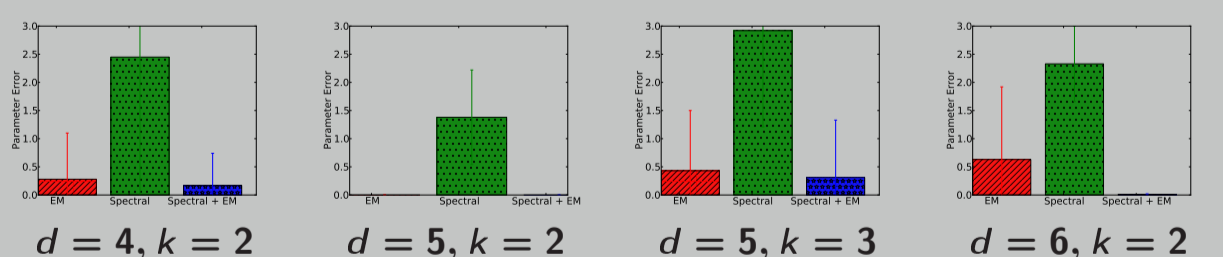
[5]: Anandkumar, Ge, Hsu, Kakade, Telgarsky, 2012.

Experiments

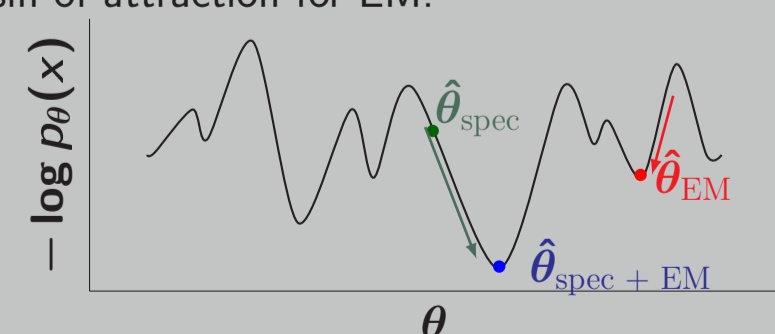
- With finite samples, Spectral Experts seems to find parameters that sufficiently separate components that EM initialized with these parameters recovers true parameters more often than EM with random initializations.
- In this example, $y = \beta^T [1, t, t^4, t^7]^T + \epsilon$. $k = 3, d = 4, n = 10^5$,



- Below are parameter errors averaged over 10 initializations on 10 different simulated datasets with the specified parameter configurations,



- Log-likelihood cartoon:** It seems that our parameter estimates fall in the right basin of attraction for EM.



Future Work

- How can we handle other discriminative models?
 - Non-linear link functions (hidden variable logistic regression).
 - Dependencies between h and x (mixture of experts).